Quantum computation as trajectory monitoring requires only one qubit in answer register in quantum phase estimation

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Abstract

Quantum computation scheme developed using classical apparatuses has been extended to systems with two qubits. To obtain controlled NOT operation, we set an interaction between two bits, control bit and target bit, with an oscillation on the target bit. Condition of controlled NOT operation simultaneously determines the operation time, interaction strength and oscillation magnitude. Subsequently, we demonstrate that, in quantum phase estimation algorithm, our system — with only one qubit in answer register — requires no iteration. Classical mechanical systems we apply do not share the same vulnerabilities as quantum systems. Moreover, our system does not require numerous measurements to determine most probable value of the system output. These features provide compelling advantages for such computational systems.

Contribution of the Paper: Our work shows a concrete method to obtain controlled NOT operation between two qubits. The result provides a method for assembling classical apparatuses to construct universal gates in quantum computers.

Keywords: quantum computation, universal gates, quantum phase estimation, equal time feedback law

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Nomenclature

ΔV	time dependent perturbation
ΔV_H	perturbation to generate Hadamard gate operation
$\lvert D\rangle$	the first excited state
$\lvert U\rangle$	the lowest energy state
λ	eigen value
$\mathsf{C_U}$	controlled unitary gate
CNOT	controlled NOT gate
H	Hadamard gate
P_ϕ	phase gate
U	unitary matrix
ω_R	frequency parameter of time dependent perturbation ΔV
ψ	wave function
θ	parametrization of λ
ε	strength of interaction V^{int}
a_C	strength of time dependent perturbation ΔV
H_0	free Hamiltonian
H_R	system parameter corresponding to Planck's constant
m	mass of a particle or strength of control input cost
n_{acn}	integer to determine a_C
n_{evn}	even integer to determine T_{CNOT} and ε
n_{odd}	odd integer to determine T_{CNOT} and ε
S	half width of wall potential

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T_{H} operating time of Hadamard gate

T_{CNOT} operating time of controlled NOT gate

u feedback input

V potential or control cost function

V^{int} interaction between two qubits

V_0 wall potential height

x state variable

$*_{\mathsf{C}}$ operation on control bit

$*_{\mathsf{T}}$ operation on target bit

1. INTRODUCTION

The development of quantum computers, which are capable of performing computations significantly faster than conventional classical computers, is accelerating rapidly. Calculations are usually iterative processes. We expect that the number of iterations can be significantly reduced [1] by utilizing the characteristics of "quantum." However, this quantumness introduces two primary challenges. The first is the fragility of quantum systems. Measurement destroys quantum superposition owing to "back-action." The second problem is that the system is essentially probabilistic. Obtaining definite numerical results requires multiple measurements.

On the contrary, classical systems are robust. We encounter no back-action in controlling classical systems. Definite value is obtained in only one measurement. Thus, if quantum computers using classical apparatuses are realized, they are positioned as one of breakthroughs towards scalable quantum computer. We have analyzed such systems for one qubit [2, 3, 4, 5, 6]. Specifically, two points have been emphasized. The first is that quantum computer with classical apparatuses allows to use macroscopic phenomenological laws, as provided by state equation, Eq.(1) in Section **3** in this paper. Secondly, feedback with equal time input-output relation, Eq.(3) in the Section **3** in this paper, provides classical systems quantum behavior. Therefore, monitoring the trajectory of the state x provides information on the weighting factor of eigenstates in quantum superposition, of the systems.

However, the former researches [2, 3, 4, 5, 6] on quantum computation using classical apparatuses are restricted to phase gate P_ϕ and Hadamard gate H for one qubit. It is known [1] that arbitrary gates used for computation are constructed from elementary gates, such as P_ϕ, H, and controlled NOT(CNOT). In this paper, to construct this CNOT gate, results of the former researches are extended to systems with two qubits. Subsequently, a scheme of quantum phase estimation with only one qubit in AnswerRegister is clarified. In terms of actual hardware, our N-bit computer consists of the same N control systems. This report is positioned as a simulation of such a hardware system.

The remainder of this paper is organized as follows. Section **2** presents a brief historical overview of quantum computation. In Section **3**, a use of classical apparatuses in quantum computing is reviewed. Based on the formulation, we first demonstrate the construction of universal gates in Section **4**. Subsequently in Section **5**, we illustrate with an example that a significant reduction in qubit number in quantum phase estimation algorithm is achieved. Summary and discussion are presented in Section **6**.

2. BACKGROUND

In verifying "many-world interpretation" of quantum mechanics, Deutsch [7, 8] found that it is possible to use superposition principle in calculation process. Subsequently, it has become a common understanding among computer science researchers that with just 50 qubits, "quantum computation" can approach the memory capacity of a supercomputer according to "Moore law" [1]. In quantum nature, various qubits develop in time, such that they are correlated or entangled. The most important feature of quantum superposition is conducted by the entanglement. Algorithms that fully use the superposition principle were typically developed by Shor [9, 10] and Grover [11, 12]. Quantum transcendence has been demonstrated on hardware [13], and various algorithms [14] have been continuously proposed and are being demonstrated in response.

However, quantum systems are fragile. Error correction in quantum computation is necessary. To ensure fault-tolerance, the vulnerability requires unrealistic numbers of qubits [15, 16]. For example, there are studies that estimate that prime factorization of a number with 2048 bits requires 20 million or 6 billion qubits [17, 18, 19]. The practical construction of quantum computer hardware presents substantial challenges due to peripheral system requirements.

In a gated quantum computer utilizing binary spin quantization, the logic of quantum linear superposition is highly vulnerable to disruption by thermal noise. Operation is only possible under extremely low-temperature conditions. We need large-scale cooling apparatus around it. Quantum computers that use light do not require an extremely low temperature environment. However, light travels straight. It is challenging to integrate such a system onto a small-scale chip. However, for this light type, it was recently reported that researchers successfully developed fault-tolerant quantum computer [20]. In the situation described above, we expect a concept "NISQ"(Noisy Intermediate-Scale Quantum) [21, 22, 23] with at most ∼100 qubits. For example, Google's random quantum circuit sampling [13] has revealed certainty of quantum supremacy and existence of hardware with 53-qubits.

Researchers across various fields are beginning to make full-scale use of "quantum" to accelerate computation. The calculations are conducted on conventional computers. However, many algorithms are developed especially for NISQ quantum hardware. In AI(artificial intelligence) technologies, algorithm by Harrow, Hassidim and Lloyd [24] that is used in linear equations solver [25, 26, 27], optimization [28, 29, 30] by Grover [11, 12] and quantum circuit learning [31] are tried. Appropriately setting initial wave function [32] is important in quantum phase estimation, especially in quantum chemistry. To determine ground state of quantum many-body systems, variational quantum eigensolver algorithm [33] is employed. In addition, further applications are emerging — one example is the quantum simulator studied in [34]. However, "scalable fault-tolerant quantum computer" remains a goal among quantum computer researchers.

3. QUANTUM COMPUTATION AS TRAJECTORY MONITORING

For simplicity, one dimensional system is selected for analysis. We wrote "for simplicity." However, if we can find one dimensional state variable as a suitable candidate of quantum computation, the state equation Eq.(1) is not merely written for simplicity; rather, it accurately represents the behavior of the real world. Mass point moving in one dimension in Newton mechanics is one of examples of such real world. Furthermore, temperature control systems can be described by Eq.(1):

$$\dot{x} = g(x)u + F(x). \tag{1}$$

With $g(x) = 1$ and $F(x) = 0$, Eq.(1) describes mass point moving in a straight line in one dimension, where x represents linear coordinate and u is the velocity of the point. In addition, Eq.(1) can describe a temperature control system, where x represents temperature, u is the flow rate, $g(x)$ is the combustion rate, and $F(x)$ is the heat dissipation. The control specification requires minimizing the time integration of difference $L(x,u) \equiv \frac{m}{2}u^2 - V(x)$ which accounts for the costs of manipulation, $\frac{m}{2}u^2$, and control errors, $V(x)$. Consider state equation Eq.(1) as constraint, and introduce a new dynamical variable μ corresponding to the constraint. These conditions enable us to express the system in terms of minizing time integral of the sum of L and $\mu\cdot(g(x)u+F(x)-\dot{x})$. Dirac's recipe [35] of constrained dynamics demonstrates how to quantize the system to finally obtain quantum mechanical wave equation [36, 37],

$$\begin{aligned} iH_R\frac{\partial\psi}{\partial t} &= -\frac{{H_R}^2}{2m}\left(\overline{g\frac{\partial}{\partial x}}\right)^2\psi \\ &\quad +V\psi - iH_R\overline{F\frac{\partial}{\partial x}}\psi, \end{aligned} \tag{2}$$

with a definition $\overline{\omega\sigma} = \frac{\omega\sigma+\sigma\omega}{2}$. In Eq.(2), H_R is a design constant that characterizes amplitude of quantum fluctuation of the system. Tunneling phenomena are not rare in classical when appropriately shaped. Passing through a potential barrier is only a deviation from the control specification, given by $V(x)$, which is commonly observed in control process. Setting $g = 1$, $F = 0$ and $H_R(= \hbar \equiv \frac{h}{2\pi})$, Planck's constant $h = 6.62607015 \times 10^{-34}$[1], provides the conventional Newton mechanics and quantum mechanics, Schrödinger equation, of mass particle. The use of a polar coordinate representation, $\psi = |\psi|e^{\frac{iS}{H_R}}$, allows us to express the fluctuation in a form of a potential $V^q[|\psi|]$ [38, 39, 40]. With $V + V^q$ instead of V, we finally obtain a generalized Hamilton-Jacobi equation for value function S, from which a feedback law,

$$u = \frac{g}{m}\frac{\partial S}{\partial x} = \frac{g^2 H_R}{2im}\frac{\psi^*\frac{\partial\psi}{\partial x} - \psi\frac{\partial\psi^*}{\partial x}}{|\psi|^2} \tag{3}$$

is calculated. Notice the existence of $|\psi|^2$ that can disappear in the denominator of Eq.(3). However, at least intuitively, it does not pose any difficulties. No particle reaches such point, because $|\psi(x;t)|^2$ is the probability that the particle exists at point x at time t. Detailed mathematical analysis on this point has been conducted [41].

The aforementioned framework can be adopted to various quantum systems, by the designer's discretion. In the following, we select one dimensional mass particle under wall potential $V(x)$, with height V_0, width $2S$.

4. CONSTRUCTING TWO-QUBITS GATES FOR UNIVERSAL GATES

First, we provide a brief review of the construction of the phase gate P_ϕ and Hadamard gate H for a single qubit [3]. Subsequently, CNOT gate for two qubits is defined.

Let H_0 be a free Hamiltonian, $H_0 = \frac{mu^2}{2} + V(x)$. Eigenstates $|U\rangle$ and $|D\rangle$ are characterized by $H_0\phi_X(x) = E_X\phi_X(x)$ $(X = \{U, D\})$, with eigen energy E_X and eigen function $\phi_X(x)$. We correspond the lowest energy state to U and the first excited state to D. Although we do not focus on spin systems, U figuratively indicates up spin and D indicates down spin. In the following, we proceed calculation by neglecting the effects of excitation to higher energy states.

4.1. Phase gate P_ϕ and Hadamard gate H for one qubit

Free time development provides a phase gate. Initial wave function $\psi(0) = \alpha|U\rangle + \beta|D\rangle$ develops in time freely into

$$\psi(T_\phi) = e^{\frac{E_U}{iH_R}T_\phi}(\alpha|U\rangle + e^{\frac{E_D-E_U}{iH_R}T_\phi}\beta|D\rangle), \tag{4}$$

where P_ϕ gate with

$$\phi = -\frac{E_D - E_U}{H_R}T_\phi, \tag{5}$$

[1] In this paper, we apply MKS unit.

is realized. In Hadamard gate H, operation is performed during the period T_H when an appropriate perturbation [3] $\Delta V_H(x)$ works. The system evolves with time, reaching its state at $t = T_H$

$$\alpha |U\rangle + \beta |D\rangle \mapsto \frac{\alpha+\beta}{\sqrt{2}} |U\rangle + \frac{\alpha-\beta}{\sqrt{2}} |D\rangle . \tag{6}$$

4.2. Controlled NOT gate CNOT *for two* qubit*s*

Now, let us discuss two qubits. We extend the results in the former researches on one qubit to provide concrete formulation for constructing CNOT gate for two qubits. Assume we prepare a system with two qubits

$$\psi(0) = \alpha |UU\rangle + \beta |UD\rangle + \gamma |DU\rangle + \delta |DD\rangle , \tag{7}$$

where we call the first qubit control bit and the second target bit. Consider a unitary matrix

$$\mathsf{U} = \begin{bmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{bmatrix}. \tag{8}$$

Controlled unitary operation $\mathsf{C_U}$ makes initial wave $\psi(0)$ develop in time to provide

$$\psi(0) = \begin{bmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{bmatrix} \mapsto \begin{bmatrix} \alpha \\ \beta \\ U_{11}\gamma + U_{12}\delta \\ U_{21}\gamma + U_{22}\delta \end{bmatrix} = \mathsf{C_U}\psi(0). \tag{9}$$

Assuming

$$\mathsf{U} = \mathsf{X} \equiv \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \tag{10}$$

$\mathsf{C_U}$ provides controlled NOT gate, one of the universal gates, $\mathsf{CNOT} = \mathsf{C_X}$. This CNOT works in a way that the target bit q_{Target} converted from $|U\rangle$ to $|D\rangle$ or $|D\rangle$ to $|U\rangle$ only if the control bit $q_{Control}$ is $|D\rangle$. It is not an easy task to find candidate physical systems that work in such a convenient way. To provied CNOT, we introduce an interaction between two qubits

$$\begin{aligned} V^{int}(x_{Control}, x_{Target}) \quad &= \\ &\epsilon \times p(x_{Control}) \times p(x_{Target}), \end{aligned} \tag{11}$$

where $p(x)$ is an even function in x. Simultaneously, on the target bit, we set a time dependent perturbation,

$$\begin{aligned} \Delta V(x_{Target}; t) \quad &= \\ &a_C \cdot \cos(\omega_R t) \cdot \left(-iH_R \frac{\partial}{\partial x_{Target}} \right). \end{aligned} \tag{12}$$

In Eq.(12), a_C and ω_R are determined by Eq.(20) and Eq.(14), respectively.

Let the initial wave function be expressed as Eq.(7). The condition that CNOT gate operation is performed at time T_{CNOT},

$$\begin{aligned} \psi(T_{\mathsf{CNOT}}) \quad = \quad & \alpha |UU\rangle + \beta |UD\rangle \\ & +\delta |DU\rangle + \gamma |DD\rangle , \end{aligned} \tag{13}$$

provides values of the parameters ε Ca_C Cω_R. Let us define $E_X \equiv \langle X|\, H_0 \,|X\rangle$, $p_X \equiv \langle X|\, p(X) \,|X\rangle$ for $X = U, D$, $\Delta E \equiv E_D - E_U$, $\Delta p \equiv p_D - p_U$, $H_R\omega_X \equiv \Delta E + \varepsilon p_X \Delta p$, $X_{UdD} \equiv \int dx \phi_U(x) \frac{d}{dx}\phi_D(x)$ and $\Omega \equiv a_C X_{UdD}$. When we assume

$$\omega_R = \omega_D, \tag{14}$$

time integration of Schrödinger equation is an exercise often shown in the elementary textbooks. We obtain

$$\begin{aligned} \psi(T_{\mathsf{CNOT}}) \quad = \quad & e^{\frac{E_{UU}T_{\mathsf{CNOT}}}{iH_R}} \alpha |UU\rangle \\ & + e^{\frac{E_{UD}T_{\mathsf{CNOT}}}{iH_R}} \beta |UD\rangle \\ & + (\gamma \cos \frac{\Omega T_{\mathsf{CNOT}}}{2} - \delta \sin \frac{\Omega T_{\mathsf{CNOT}}}{2}) \\ & e^{\frac{E_{DU}T_{\mathsf{CNOT}}}{iH_R}} |DU\rangle \\ & + (\gamma \sin \frac{\Omega T_{\mathsf{CNOT}}}{2} + \delta \cos \frac{\Omega T_{\mathsf{CNOT}}}{2}) \\ & e^{\frac{E_{DD}T_{\mathsf{CNOT}}}{iH_R}} |DD\rangle . \end{aligned} \tag{15}$$

The wave function Eq.(15) becomes the same as that Eq.(13) except a common phase $-\frac{E_{UD}T_{\mathsf{CNOT}}}{H_R}$, when we assume

$$\sin \frac{\Omega T_{\mathsf{CNOT}}}{2} = -1, \tag{16}$$

$$e^{\frac{(E_{UU}-E_{UD})T_{\mathsf{CNOT}}}{iH_R}} = 1, \tag{17}$$

$$e^{\frac{(E_{DD}-E_{UD})T_{\mathsf{CNOT}}}{iH_R}} = -1. \tag{18}$$

With integer n_{acn} and even/odd integer n_{evn}/n_{odd}, parameters

$$T_{\mathsf{CNOT}} = \frac{\pi H_R (n_{evn} p_D + n_{odd} p_U)}{\Delta E \Delta p}, \tag{19}$$

$$a_C = \frac{(3 + 4 \cdot (n_{acn} - 1))\, \pi}{X_{UdD}} \frac{1}{T_{\mathsf{CNOT}}}, \tag{20}$$

$$\epsilon = -\frac{(n_{odd} + n_{evn})\Delta E}{\Delta p (n_{evn} p_D + n_{odd} p_U)} \tag{21}$$

satisfy Eqs.(16), (17) and (18).

4.3. Assembling universal gates to give controlled unitary gate

Combination of these P_ϕ, H, CNOT gates allows us to obtain various gates. Here, we demonstrate a standard way to assemble universal gates to provide controlled unitary gate, $\mathsf{C_U}$. First, express rotation gates around y, z axes as $\mathsf{R_y}(\phi) \equiv e^{-i\frac{\phi}{2}} \mathsf{P}_{\frac{\pi}{2}} \mathsf{H} \mathsf{P}_\phi \mathsf{H} \mathsf{P}_{-\frac{\pi}{2}}$, $\mathsf{R_z}(\phi) \equiv e^{-i\frac{\phi}{2}} \mathsf{P}_\phi$. Subsequently, define $\mathsf{A} \equiv \mathsf{R_y}(\frac{\alpha-\gamma}{2})$, $\mathsf{B} \equiv \mathsf{R_y}(-\frac{\alpha+\gamma}{2})\mathsf{R_z}(-\frac{\beta}{2})$, $\mathsf{C} \equiv \mathsf{R_z}(\frac{\beta}{2})\mathsf{R_y}(\gamma)$. We can obtain 4 parameters α, β, γ and

θ to provide an arbitrary unitary matrix $\mathsf{U} \in 2 \times 2$ in a form

$$\mathsf{U} = \mathsf{AXBXCP}_\theta. \tag{22}$$

According to Eq.(10) in Eq.(22), we assemble controlled unitary gate $\mathsf{C_U}$ as follows,

$$\mathsf{C_U} = \mathsf{A_T} \cdot \mathsf{CNOT} \cdot \mathsf{B_T} \cdot \mathsf{CNOT} \cdot \mathsf{C_T C_{P_\theta}}, \tag{23}$$

$$\mathsf{C_{P_\theta}} = \mathsf{P}_{\frac{\theta}{2}\mathsf{C}} \cdot \mathsf{P}_{\frac{-\theta}{2}\mathsf{T}} \cdot \mathsf{CNOT} \cdot \mathsf{P}_{\frac{\theta}{2}\mathsf{T}}, \tag{24}$$

where the suffices C and T mean that the operation is on $q_{Control}$ and q_{Target}, respectively.

Examples are provided in the following. We apply a system with mass $m = 1$, wall potential height $V_0 = 1$, width $2S = 2(S = 1)$. Quantization constant is chosen as $H_R = 0.5$ to guarantee two-level structure of energy [2].

First, we present an example of the perturbation as $\Delta V_H(x) = 0$ for $|x| > S$ and

$$\Delta V_H(x) = a_P \left(x^2 - S^2\right) + b_P \sin \frac{\pi x}{2S}, \tag{25}$$

for $|x| \leq S$. The time T_H is calculated as a function of a_P and b_P. We can construct H gate by using time dependent perturbation, ΔV in Eq.(12), instead of ΔV_H in Eq.(25).

Secondly, we provide CNOT by setting $p(x) = x^2$ in Eq.(11). Let us randomly assume an initial wave function

$$\begin{aligned}\psi &= 0.45e^{1.80i} |UU\rangle + 0.38e^{-1.49i} |UD\rangle \\ &+0.48e^{1.10i} |DU\rangle + 0.65e^{2.78i} |DD\rangle\end{aligned} \tag{26}$$

Values $n_{evn} = 100$,$n_{odd} = -101$,$n_{acn} = 0$ yield $a_C = -0.01$,$\varepsilon = 0.02$,$T_{CNOT} = 341.72$ and at $t = T_{CNOT}$,

$$\begin{aligned}\psi &= 0.53e^{-0.70i} |UU\rangle + 0.25e^{-2.92i} |UD\rangle \\ &+0.65e^{1.44i} |DU\rangle + 0.48e^{-0.23i} |DD\rangle\end{aligned} \tag{27}$$

that is not a CNOT operation result of Eq.(26). The reason why we failed to obtain the appropriate result in Eq.(27) is due to the discrepancy $\omega_D = 0.93$ and $\omega_U = 0.92$ being significantly small, 0.01 of the mean. Thus, we retried to assume $n_{evn} = 80$, $n_{odd} = -101$,$n_{acn} = 0$ to obtain larger difference between $\omega_U = 1.07$ and $\omega_D = 1.36$, ratio 0.23 of the mean. Parameters are calculated as $a_C = -0.01$,$\varepsilon = 0.62$,$T_{CNOT} = 233.83$. At $t = T_{CNOT}$ we obtain

$$\begin{aligned}\psi &= 0.45e^{1.23i} |UU\rangle + 0.38e^{-2.08i} |UD\rangle \\ &+0.65e^{2.28i} |DU\rangle + 0.48e^{0.60i} |DD\rangle ,\end{aligned} \tag{28}$$

that well approximates CNOT operation result of Eq.(26).

5. QUANTUM PHASE ESTIMATION

Let us consider 2^{m_B} dimensional unitary matrix

$$\mathsf{U} \in (2^{m_B} \times 2^{m_B}) \tag{29}$$

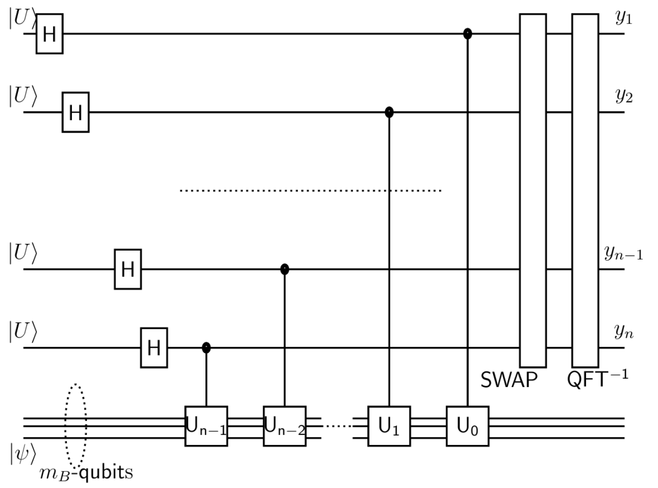

Figure 1: Quantum phase estimation with n qubits to calculate eigenvalue $\lambda = e^{2\pi i\theta}$ in n-th order in Eq.(31). We defined $\mathsf{U_k} \equiv \mathsf{U}^{2^k} (k = 0, 1, \cdots, n-1)$, especially $\mathsf{U_0} = \mathsf{U}^{2^0} = \mathsf{U}^1 = \mathsf{U}$.

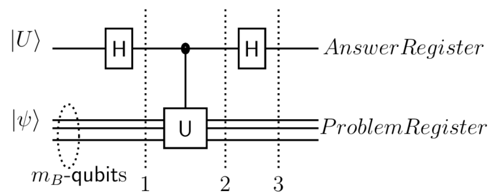

Figure 2: Hadamard test circuit.

and assume that we know one of its eigenvectors

$$\psi \in (2^{m_B} \times 1). \tag{30}$$

As shown in Fig.**1**, quantum phase estimation algorithm finds the corresponding eigenvalue $\lambda = e^{2\pi i\theta}$ in a form

$$2^n\theta = y_1 2^0 + y_2 2^1 + y_3 2^2 + \cdots , \tag{31}$$

where y_1,$\cdots$,y_n are the output of inverse quantum fourier transformation QFT^{-1}. It is evident that, in principle, the conventional algorithm requires infinite number of qubits as accuracy requirements become increasingly stringent. [2] At a first glance, a simpler "Hadamard test" circuit shown in Fig.**2** can be used to find the eigenvalue λ. In particular, at time 2, our wave function has information on λ in a register that we measure to obtain the calculation result. In Fig.**2**, we refer to the register as AnswerRegister. A register that stores the known eigenvector ψ is called ProblemRegister. Initial wave function $|U\rangle \otimes \psi$ develops in time to $\Psi_{(1)} = \mathsf{H}|U\rangle \otimes \psi = \frac{|U\rangle + |D\rangle}{\sqrt{2}} \otimes \psi$. The unitary matrix U operates on the state ψ of the ProblemRegister, only when the state of

[2] The problem can be solved by using iterative QPE [42]. Our scheme needs no iteration.

the qubit on the AnswerRegister is $|D\rangle$. Thus, the controlled unitary operation $\mathsf{C_U}$ results in

$$\Psi_{(2)} = \frac{1}{\sqrt{2}}\left(|U\rangle \otimes \psi + |D\rangle \otimes \mathsf{U}\psi\right). \tag{32}$$

We expect that the measurement on Eq.(32) provides some information on λ owing to the eigenvalue equation $\mathsf{U}\psi = \lambda\psi$. However, the result is practically useless, as it yields only probabilities $Pr_{|U\rangle} = \frac{1}{2}$ and $Pr_{|D\rangle} = \frac{|\lambda|^2}{2} = \frac{1}{2}$. Further sophisticated approach can involve measurement on

$$\begin{aligned}\Psi_{(3)} &= \frac{1}{\sqrt{2}}\left(\frac{|U\rangle + |D\rangle}{\sqrt{2}} \otimes \psi + \frac{|U\rangle - |D\rangle}{\sqrt{2}} \otimes \mathsf{U}\psi\right) \\ &= \left(\frac{1+\lambda}{2}|U\rangle + \frac{1-\lambda}{2}|D\rangle\right) \otimes \psi. \end{aligned} \tag{33}$$

Real part λ_R =Reλ is obtained by measuring $\Psi_{(3)}$: $Pr_{|U\rangle} = \left|\frac{1+\lambda}{2}\right|^2 = \frac{1+\lambda_R}{2}$. Moreover, a simple modification (adding an S gate between H and Controlled-U) provides the imaginary part. However, the conventional quantum computation needs numerous times of measurements. On the contrary, using only one measurement, our proposed method determines the eigenvalue λ at time 2 in the simple circuit of "Hadamard test."

The wave function develops in time through various combinations of universal gates [1] during $\mathsf{C_U}$ gate. The wave function itself does not inherently contain λ. Moreover, it has no awareness that it will eventually take the simple form of Eq.(32) after the $\mathsf{C_U}$ operation. Can we determine λ value by comparing particle motion by the wave function, which inherently lacks information on λ, under explicit form Eq.(32)? Therefore, this, we express $\Psi_{(2)}$ after controlled unitary operation, that ends at $t = T_{CU}$, in coordinate representation

$$\begin{aligned}\Psi_{(2)}(x_1, \vec{x}_2, t) &= \frac{1}{\sqrt{2}}[e^{\frac{(t-T_{CU})E_U}{iH_R}}\phi_U(x_1) \\ &+ e^{\frac{(t-T_{CU})E_D}{iH_R}}\lambda\phi_D(x_1)]\psi_2(\vec{x}_2, t). \end{aligned} \tag{34}$$

In Eq.(34), $x_1 \in \mathsf{R}$ and $\vec{x}_2 \in \mathsf{R}^{m_B}$ are the particle coordinate of one qubit in AnswerRegister and that of m_B-qubits in ProblemRegister, respectively. Exponential functions $e^{\frac{(t-T_{CU})E_X}{H_R}}$, $(X = \{U, D\})$ represent free motion after $\mathsf{C_U}$ gate operation. The feedback $u(x_1, \vec{x}_2; \lambda)$ is computed by Eq.(3) using the above expression Eq.(34). Comparing this $u(x_1, \vec{x}_2; \lambda)$ with measured feedback value reveals λ.

A numerical example is provided below for $m_B = 1$ or $\psi \in \mathsf{R}^2$. As shown in Eqs.(22), (23) and (24), we can prepare arbitrary controlled unitary operation $\mathsf{C_U}$ by combining CNOT and unitary matrix U [1]. The gate CNOT can be calculated using Eqs.(19), (20) and (21).

Assume an arbitrary unitary matrix

$$\mathsf{U} = \begin{bmatrix} 0.3771 + 0.5947i & -0.6814 + 0.1995i \\ 0.1313 + 0.6978i & 0.7010 + 0.0664i \end{bmatrix}. \tag{35}$$

The parameters required to implement $\mathsf{C_U}$ gate in a form Eq.(23) are calculated as $\alpha = 1.2101$, $\beta = -2.3205$, $\gamma = -0.5312$, $\theta = 1.1001$. Assume that one of the eigenvector is provided as follows

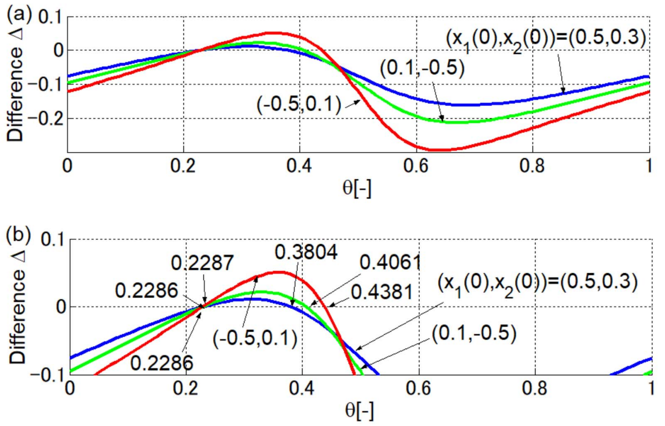

Figure 3: Difference Δ as a function of θ for three different pairs of initial conditions $(x_1(0), x_2(0))$. Initial coordinates $(x_1(0), x_2(0))$ are $(0.5, 0.3)$ for blue line, $(0.1, -0.5)$ for green line and $(-0.5, 0.1)$ for red line, respectively. Figure (a) shows overall variations of Δ relative to θ. While (b) shows in detail the points θ where these Δ cross the line $\Delta = 0$.

$$\psi = \begin{bmatrix} 0.8367 \\ 0.4060 - 0.3677i \end{bmatrix}. \tag{36}$$

Let Δ provide difference between $u(x_1, x_2; \lambda)$ and measured value. Figure 3 shows the difference Δ as a function of $\theta \in [0, 1]$, where θ parametrizes $\lambda = e^{2\pi\theta i}$. Among infinite pairs of initial coordinate $(x_1(0), x_2(0))$, three pairs are tried: blue line for $(0.5, 0.3)$, green for $(0.1, -0.5)$ and red for $(-0.5, 0.1)$. The calculation along blue line provides two solutions: $\theta = 0.2287$ and 0.3804. The green line crosses $\Delta = 0$ at $\theta = 0.2286$ and 0.4061. Profile of the third, red, line indicates $\theta = 0.2286$ and 0.4381. The θ value that all these three pairs commonly indicates that the difference Δ approximately assumes zero is $\theta = 0.2286$. We can verify that this value correctly leads to the eigenvalue $\lambda = e^{i2\pi\times 0.2286} = 0.1342 + 0.991i$.

The answer $\theta = 0.2286\cdots$ is expressed as infinite series $\theta = 0.0011\cdots$ in binary number. Therefore, the conventional methods, as illustrated in Fig.1, require infinite number of qubits. It should be noted that the result is probabilistic in nature, necessitating repeated measurements to obtain reliable outcomes.

6. CONCLUSIONS

We presented how the controlled-NOT(CNOT) gate, a fundamental universal quantum gate for two qubits, is physically realized in classical systems through carefully designed feedback mechanisms. Feedback generation during gate operation does not require a time integration task. Moreover, we found that quantum phase estimation algorithm, interpreted according to trajectory monitoring, requires only one qubit in AnswerRegister.

Our immediate task is to examine the perturbation parameters in Eqs.(11), (12), and (25) such that the system evolves within the linear space spanned by combinations of $|U\rangle$ and $|D\rangle$. Subsequently, applying the proposed systems to real problems, including Shor, Grover, ... algorithms, is to be followed. Our quantum computing system does not require large peripherals. Thus, the installation of the system to 'edge' or machine terminal represents an interesting research topic.

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